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ON FINDIND INTEGER SOLUTIONS TO THE NON-HOMOGENEOUS TERNARY QUINTIC DIOPHANTINE EQUATION

$$3(x^2+y^2)-5xy=15z^5$$

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ABSTRACT:

This Paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous Quintic Diophantine equation with three unknowns given by $3(x^2+y^2)-5xy=15z^5$. Various sets of distinct integer solutions to the considered quintic equation are studied through employing the linear transformation $x=u+v, y=u-v, (u\neq v\neq 0)$ and applying the method of factorization.

KEYWORDS: Ternary quintic, Non-homogeneous quintic , Integer solutions

INTRODUCTION:

The non-homogeneous ternary Quintic Diophantine equation offers an unlimited field for research due to their variety [1-4]. In particular, one may refer [5-16] for quintic equations with three and five unknowns. This communication concerns with yet another interesting ternary quintic equation $3(x^2+y^2)-5xy=15z^5$ is analysed for its non-zero distinct integer solutions through different methods.

METHOD OF ANALYSIS:

The Non-Homogeneous Ternary Quintic Diophantine equation to be solved for non-zero distinct integral solution is

$$3(x^2+y^2)-5xy=15z^5 \quad (1)$$

To start with, observe that (1) is satisfied by the following integer triples (x,y,z):(5,3,1) (-5,-3,1),(3,5,1),(-3,-5,1), $(54\alpha^{5k},27\alpha^{5k},3\alpha^{2k})$

$(15^3k(3k^2-5k+3)^2,15^3(3k^2-5k+3)^2,15(3k^2-5k+3))$

However, there are other sets of integer solutions to (1) that are illustrated below:

ILLUSTRATION 1:

Introduction of the linear transformation

$$x=u+v, y=u-v, u\neq v\neq 0 \quad (2)$$

In (1), it becomes

$$u^2+11v^2=15z^5 \quad (3)$$

The above equation is solved for u, v and z through different methods and using (2), the values of **x** and y satisfying (1), are obtained which are given below

METHOD 1:

After performing a few calculations, it is observed that (3) is satisfied by,

$$u=15^3m(m^2+11n^2)^2$$
$$v=15^3n(m^2+11n^2)^2$$
$$z=15(m^2+11n^2)$$

In view of (2), the corresponding integer solutions to (1) are found to be

$$x = 15^3 (m+n)[(m^2+11n^2)^2]$$
$$y = 15^3 (m-n)[(m^2+11n^2)^2] \qquad (4)$$
$$z = 15(m^2+11n^2)$$

ILLUSTRATION 3:

Assume $z = a^2 + 11b^2$ (5)

Case (i):

Write 15 as

$$15 = (2 + i\sqrt{11})(2 - i\sqrt{11}) \qquad (6)$$

Using (5) and (6) in (3) and employing the method of factorization, define

$$(u + i\sqrt{11}v) = (2 + i\sqrt{11})(a + i\sqrt{11}b)^5$$

Equating the real and imaginary parts, we get

$$u = 2a^5 - 220a^3b^2 + 1210b^4 - 55a^4b + 1210a^2b^3 - 1331b^5$$
$$v = a^5 + 10a^4b - 220a^2b^3 + 242b^5 - 110a^3b^2 + 605ab^4$$

In view (2), we obtain

$$\left.\begin{aligned} x &= 3a^5 - 330a^3b^2 - 45a^4b + 990a^2b^3 + 1815ab^4 - 1089b^5 \\ y &= a^5 - 65a^4b - 110a^3b^2 + 1430a^2b^3 + 605ab^4 - 1573b^5 \end{aligned}\right\} \qquad (7)$$

Thus (5) and (7) represent the integer solutions to (1).

Case (ii):

Write 15 as

$$15 = \frac{(7 + i\sqrt{11})(7 - i\sqrt{11})}{4} \qquad (8)$$

Using (5) and (8) in (3) and applying the method of factorization , define

$$(u + i\sqrt{11}v) = \frac{(7 + i\sqrt{11})}{2}(a + i\sqrt{11}b)^5$$

Equating the real and imaginary parts, we get

$$u = \frac{1}{2}\left[7(a^5 - 110a^3b^2 + 605ab^4) - 11(5a^4b - 110a^2b^3 + 121b^5)\right]$$
$$v = \frac{1}{2}\left[(a^5 - 110a^3b^2 + 605ab^4) + 7(5a^4b - 110a^2b^3 + 121b^5)\right]$$

In view (2), we obtain

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$$\left.\begin{aligned} x &= 4a^5 - 440a^3b^2 + 2420ab^4 - 10a^4b - 220a^2b^3 - 242b^5 \\ y &= 3a^5 - 330a^3b^2 + 1815ab^4 - 45a^4b + 990a^2b^3 - 1089b^5 \end{aligned}\right\} \quad (9)$$

Thus (5) and (9) represents the integer solution to (1).

METHOD III:

Equation (3) can be written as

$$u^2 + 11v^2 = 15z^5 * 1 \quad (10)$$

Write 1 on the R.H.S. of (10) as

$$1 = \frac{(1 + i3\sqrt{11})(1 - i3\sqrt{11})}{100} \quad (11)$$

Using (6), (6) & (11) in (10) and utilizing the method of factorization, define

$$(u + i\sqrt{11}v) = (2 + i\sqrt{11})(a + i\sqrt{11}b)^5 \left[\frac{(1 + i3\sqrt{11})}{10}\right]$$

Equating the real and imaginary parts, we get

$$u = \frac{1}{10}\left\{-31(a^5 - 110a^3b^2 + 605ab^4) - 77(5a^4b - 110a^2b^3 + 121b^5)\right\}$$

$$v = \frac{1}{10}\left\{7(a^5 - 110a^3b^2 + 605ab^4) - 31(5a^4b - 110a^2b^3 + 121b^5)\right\}$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= \frac{1}{5}\left\{-12(a^5 - 110a^3b^2 + 605ab^4) - 54(5a^4b - 110a^2b^3 + 121b^5)\right\} \\ y &= \frac{1}{5}\left\{-19(a^5 - 110a^3b^2 + 605ab^4) - 23(5a^4b - 110a^2b^3 + 121b^5)\right\} \end{aligned}\right\} \quad (12)$$

To obtain the integer solutions, replacing a by 5A and b by 5B in (5) & (12), the corresponding integer solutions of (1) are given by

$$\left.\begin{aligned} x &= -5^4\left\{12(A^5 - 110A^3B^2 + 605AB^4) + 54(5A^4B - 110A^2B^3 + 121B^5)\right\} \\ y &= -5^4\left\{19(A^5 - 110A^3B^2 + 605AB^4) + 23(5A^4B - 110A^2B^3 + 121B^5)\right\} \\ z &= 25(A^2 + 11B^2) \end{aligned}\right\} \quad (13)$$

Thus (5) & (13) represent the integer solution to (1).

NOTE 1:

The integer 1 on the R.H.S of (10) is also expressed as below:

(i) $$1=\frac{(5+i\sqrt{11})(5-i\sqrt{11})}{36}$$

(ii) $$1=\frac{\left(2\sqrt{15+\sqrt{11}}\right)(2\sqrt{15}-\sqrt{11})}{49}$$

By considering suitable combinations of integers 15 &1 from Note 1 in (3), some more sets of integer solutions to (1) are obtained.

ILLUSTRATION II:

Introduction of the linear transformations

$$x=15^3(u+v), y=15^3(u-v)\ , \qquad u\neq v\neq 0 \tag{14}$$

In (1) leads to

$$u^2+11v^2=w^5 \tag{15}$$

The above equation is solved for u, v and z through different methods and using (14), the values of x and y satisfying (1), are obtained which are illustrated below

METHOD IV:

Write 1one the R.H.S of (15) as

$$1=\frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100} \tag{16}$$

Using (5) and (16) in (14) and employing the method of factorization, define

$$(u+i\sqrt{11}v)=(a+i\sqrt{11}b)^5\left[\frac{(1+i3\sqrt{11})}{10}\right]$$

Equating the real and imaginary parts, we get

$$u=\frac{1}{10}\{1(a^5-110a^3b^2+605ab^4)-33(5a^4b-110a^2b^3+121b^5)\}$$
$$v=\frac{1}{10}\{3(a^5-110a^3b^2+605ab^4)+1(5a^4b-110a^2b^3+121b^5)\}$$

In view (13), the corresponding integer solutions to (1) are found to be

$$\begin{aligned} X &= 5^4\{2(a^5 - 110a^3b^2 + 605ab^4) - 16(5a^4b - 110a^2b^3 + 121b^5)\} \\ Y &= 5^4\{-1(a^5 - 110a^3b^2 + 605ab^4) - 17(5a^4b - 110a^2b^3 + 121b^5)\} \end{aligned} \quad (17)$$

To obtain the integer solutions, replacing a by 5A and b by 5B in (5) & (17), the corresponding integer solutions of (1) are given by

$$\left.\begin{aligned} x &= 15^3 5^4\{2(A^5 - 110A^3B^2 + 605AB^4) - 16(5A^4B - 110A^2B^3 + 121B^5)\} \\ y &= 15^3 5^4\{-1(A^5 - 110A^3B^2 + 605AB^4) - 17(5A^4B - 110A^2B^3 + 121B^5)\} \\ z &= 15W \end{aligned}\right\} \quad (18)$$

Thus (5) and (18) represents the integer solutions to (1).

CONCLUSION:

In this paper, we have presented four different methods of obtaining infinitely many non-zero distinct integer solutions of the non-homogeneous given by $3(x^2 + y^2) - 5xy = 15z^5$. To conclude, one may search for integer solutions to the other choice of non-homogeneous ternary quintic Diophantine equations along with suitable properties.

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